RATIONAL APPROXIMATION AND LAGRANGIAN INCLUSIONS

RASUL SHAFIKOV* AND ALEXANDRE SUKHOV**

Abstract. We show that any real compact surface S, except the sphere S^2 and the projective plane $\mathbb{R}P_2$, admits a pair of smooth complex-valued functions f_1, f_2 with the property that any continuous complex-valued function on S is a uniform limit of a sequence of $R_j(f_1, f_2)$, where $R_j(z_1, z_2)$ are rational functions on $\mathbb{C}^2$.

* Department of Mathematics, the University of Western Ontario, London, Ontario, N6A 5B7, Canada, e-mail: shafikov@uwo.ca. The author is partially supported by the Natural Sciences and Engineering Research Council of Canada.

**Université des Sciences et Technologies de Lille, U.F.R. de Mathématiques, 59655 Villeneuve d'Ascq, Cedex, France, e-mail: sukhov@math.univ-lille1.fr. The author is partially supported by Labex CEMPI.

1. Introduction

This work concerns approximation of continuous functions on a compact real surface by a special class of smooth functions. To illustrate this we consider the one-dimensional example first. In the space of continuous complex-valued functions on the unit circle $S^1 \subset \mathbb{C}$ let $\mathcal{R} \subset C^0(S^1)$ be the subalgebra of functions of the form $R(e^{i\theta})$, where $\theta \in [0, 2\pi]$ and $R(z)$ is a rational function on $\mathbb{C}$ with poles off S^1. It follows from the Stone-Weierstrass theorem that $\mathcal{R}$ is dense in $C^0(S^1)$. Note that by the maximum principle the subspace of polynomials in $e^{i\theta}$ is not dense in $C^0(S^1)$. We consider the case of dimension 2. Our main result is the following

Theorem 1.1. *Let S be a smooth compact real surface without boundary, and let $C^0(S)$ be the space of continuous complex-valued functions on S. There exists a pair of smooth functions $f_j : S \to \mathbb{C}$, $j = 1, 2$, such that for every function $F \in C^0(S)$ there is a sequence $\{R_n(z_1, z_2)\}$ of rational functions on $\mathbb{C}^2$ with the following properties:*

(i) *For every n the denominator of the composition $R_n(f_1, f_2)$ does not vanish on S.*
(ii) *If S is not the unit sphere S^2 and is not the projective plane $\mathbb{R}P_2$, then $\{R_n(f_1, f_2)\}$ converges to F in $C^0(S)$.*
(iii) *If $S = S^2$, then there exists a rotation τ of S^2 (depending on F) such that $\{R_n(f_1, f_2)\}$ converges to the composition $F \circ \tau$ in $C^0(S^2)$.*
(iv) *If $S = \mathbb{R}P_2$, then there exists a smooth diffeomorphism τ of $\mathbb{R}P_2$ (depending on F) such that $\{R_n(f_1, f_2)\}$ converges to the composition $F \circ \tau$ in $C^0(\mathbb{R}P_2)$.*

This result provides an affirmative answer to the question communicated to us by Nemirovski. Note that the pair f_1, f_2 is independent of F, and that rational functions in Theorem 1.1 cannot be replaced by polynomials. To see this, suppose that for a given surface S there exist continuous

MSC: 32E20,32E30,32V40,53D12. Key words: Rational convexity, polynomial convexity, Lagrangian manifold, symplectic structure, plurisubharmonic function

functions f_1, f_2 such that any continuous function on S can be approximated by polynomials in f_1 and f_2. Since $C^0(S)$ separates points on S, the map $f = (f_1, f_2) : S \to f(S) \subset \mathbb{C}^2$ is a bijection, hence a homeomorphism. By assumption, any continuous function on $f(S)$ can be approximated by holomorphic polynomials, which forces $f(S)$ to be polynomially convex in $\mathbb{C}^2$. Recall that a compact set $X \subset \mathbb{C}^2$ is *polynomially convex* if for every point $z \in \mathbb{C}^2 \setminus X$ there is a polynomial P such that $|P(z)| > \sup_{w\in X} |P(w)|$. However, no compact topological n-dimensional submanifold of $\mathbb{C}^n$ is polynomially convex, see [18, Cor. 2.3.5]), and this proves the claim.

The functions f_1, and f_2 in Theorem 1.1 will be given as the coordinate components of a singular Lagrangian (with respect to the standard symplectic form ω_{st}) embedding of S into $\mathbb{C}^2$. For example, in the simplest case of the torus $S^1 \times S^1$, we can take $f_j = e^{i\theta_j}$, $j = 1, 2$, thinking of $\theta_j \in [0, 2\pi]$ as a parametrization of each circle S^1. For an arbitrary surface we employ in Section 2 a result of Givental [10] (see also Audin [4]), who proved the existence on S of a *Lagrangian inclusion*–a local Lagrangian embedding of S into $\mathbb{C}^2$ that can have, in addition to transverse double self-intersection points, singularities that are called *open Whitney umbrellas*; furthermore, such a map is a homeomorphism near every umbrella. Moreover, one can find such an inclusion without self-intersection points, i.e., a topological embedding, with two exceptions, the sphere S^2 and the projective plane $\mathbb{R}P_2$. These two surfaces do not admit a singular Lagrangian embedding into $\mathbb{C}^2$, but can be included with transverse double points, and so one needs more functions to generate $C^0(S)$.

Although no embedding of S into $\mathbb{C}^2$ is polynomially convex, we prove in Section 3 that there exists a Lagrangian inclusion of S into $\mathbb{C}^2$ such that its image is rationally convex. A compact set X in $\mathbb{C}^n$ is called *rationally convex* if for every point $z \in \mathbb{C}^n \setminus X$ there exists a complex algebraic hypersurface passing through z and avoiding X. This is used in the proof of Theorem 1.1 which is given in Section 4.

That rational convexity is closely connected with the property of being Lagrangian became apparent from the work of Duval [6]. Duval and Sibony [8] showed that a compact n-dimensional submanifold of $\mathbb{C}^n$ is rationally convex whenever it is Lagrangian with respect to some Kähler form. It was further proved by Gayet [9] that an immersed Lagrangian submanifold in $\mathbb{C}^n$ with transverse double self-intersections is also rationally convex. This was generalized to certain nontransverse self-intersections by Duval and Gayet [7]. Interaction between Lagrangian geometry and rational convexity was recently explored by Cieliebak-Eliashberg [5] and Nemirovski-Siegel [13] using topological methods.

Acknowledgment. The authors would like to thank Stefan Nemirovski for numerous fruitful discussions, in particular for pointing out the connection between the results concerning rational convexity of Lagrangian submanifolds and the approximation theory.

2. Lagrangian embeddings and inclusions

A nondegenerate closed 2-form ω on $\mathbb{C}^2$ is called a *symplectic form.* By Darboux's theorem every symplectic form is locally equivalent to the standard form

$$\omega_{\rm st} = \frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w}) = dd^c \phi_{\rm st}, \quad \phi_{\rm st} = |z|^2 + |w|^2,$$

where (z, w), $z = x + iy$, $w = u + iv$, are complex coordinates in $\mathbb{C}^2$, and $d^c = i(\overline{\partial} - \partial)$. If a symplectic form ω is of bidegree $(1, 1)$ and strictly positive, it is called a *Kähler form.* A smooth function ϕ is called strictly plurisubharmonic if $dd^c \phi$ is strictly positive definite. It is called a

potential of ω if $dd^c\phi = \omega$. A real n-dimensional submanifold $S \subset \mathbb{C}^n$ is called *Lagrangian* with respect to ω if $\omega|_S = 0$.

It follows from Arnold [2] that a compact Lagrangian submanifold of $\mathbb{C}^n$ has zero Euler characteristic. On the other hand, according to the result of Givental [10], any compact surface admits a *Lagrangian inclusion* into $\mathbb{C}^2$ (we use the terminology introduced in Arnold [3]), i.e., a smooth map $\iota : S \to \mathbb{C}^2$ which is a local Lagrangian embedding (i.e., $\iota^*\omega_{st} = 0$) except a finite set of singular points that are either transverse double self-intersections (or simply *double points*) or the so-called *open Whitney umbrellas*. The *standard open Whitney umbrella* is the map

$$\pi : \mathbb{R}^2_{(t,s)} \ni (t,s) \mapsto \left(ts, \frac{2t^3}{3}, t^2, s\right) \in \mathbb{R}^4_{(x,u,y,v)}. \tag{1}$$

Images of the standard open Whitney umbrella under complex affine maps that preserve the symplectic form $\omega_{\rm st}$ will also be called standard umbrellas. Finally, open Whitney umbrellas are defined as images of the standard umbrella under a local symplectomorphism, i.e., a local diffeomorphism that preserves the form $\omega_{\rm st}$. If S is orientable then each inclusion satisfies the following topological identity

$$-\chi(S) + 2\cdot d - m = 0, \tag{2}$$

and if S is nonorientable, then

$$\chi(S) + 2\cdot d - m = 0 \mod 4. \tag{3}$$

Here $\chi(S)$ is the Euler characteristic of S, d is the number of double points, and m is the number of umbrella points.

In the orientable case, a double point should be counted taking into account its index, which comes from some orientation on S and the standard orientation on $\mathbb{C}^2$. In fact, according to the result of Audin [4], any combination of numbers $\chi(S)$, d, and m, for which formula (2) is valid, can be realized in a Lagrangian inclusion. In particular, if $\chi(S) \le 0$, then we may choose $d = 0$, and $m = -\chi(S)$. This means that any orientable surface, except the sphere S^2, admits a singular Lagrangian embedding (i.e., inclusion without double points), while the Whitney sphere $\mathcal{W}|_{S^2} : S^2 \to \mathbb{C}^2$, where

$$\mathcal{W} : \mathbb{R}^3 \ni (t,s,\tau) \to (t + it\tau, s + is\tau), \tag{4}$$

is a Lagrangian immersion of S^2 with one double point.

In the nonorientable case formula (3) is valid mod 4 according to [4]. Givental [10] showed that if $\chi(S) \le -2$, then in fact we may take $d = 0$, that is, all such surfaces admit a singular Lagrangian embedding into $\mathbb{C}^2$. He also gave an explicit construction of a Lagrangian inclusion of $\mathbb{R}P_2$ with two double points and one umbrella. Recently Nemirovski and Siegel [13] gave all possibilities for the number of umbrella points that may appear in a singular Lagrangian embedding of an arbitrary compact surface S. These are given by

(i) $m = -\chi(S)$ and $\chi \ne 2$, if S is orientable;
(ii) $(\chi(S), m) \ne (1,1)$ or $(0,0)$, and $m \in \{4 - 3\chi, -3\chi, -3\chi - 4, ..., \chi + 4 - 4\lfloor \chi/4 + 1\rfloor\}$, if S is nonorientable.

In particular, all nonorientable surfaces except $\mathbb{R}P_2$ admit a singular Lagrangian embedding, while Givental's inclusion of $\mathbb{R}P_2$ into $\mathbb{C}^2$ with two double points and one umbrella has the simplest possible combination of singularities.

Suppose now that $\iota : S \to \mathbb{C}^2$ is a Lagrangian inclusion with umbrella points $p_1, \dots, p_m$. Then, in a neighbourhood U_j of every p_j, there exists a symplectomorphism $\phi_j : U_0 \to U_j$ from a neighbourhood of the origin in $\mathbb{C}^2$ that maps the standard umbrella (1) to $\iota(S) \cap U_j$.

Any symplectomorphism ϕ is locally Hamiltonian. This means that in a (simply connected) neighbourhood U there exists a smooth function $h : U \to \mathbb{R}$, called the *Hamiltonian*, such that the vector field V_h, uniquely defined by the equation

$$i(V_h)\,\omega_{st} = dh, \tag{5}$$

gives the flow ϕ^h_τ on U with the property that $\phi^h_1 = \phi$. Here $i(V_h)$ is the contraction operator. Conversely, a smooth function $h : \mathbb{C}^2 \to \mathbb{R}$ with compact support defines uniquely a vector field V_h that satisfies (5). The flow of V_h generates a one parameter family of symplectomorphisms of $\mathbb{C}^2$. These symplectomorphisms are the identity outside the support of h.

Let L_j be the linear translation in $\mathbb{C}^2$ sending p_j to the origin, and let h_j be the Hamiltonian of the symplectic maps $L_j^{-1} \circ \phi_j^{-1}$ defined in a neighbourhood U_j of p_j. Let h be a smooth function on $\mathbb{C}^2$ that agrees with h_j in U_j and vanishes outside a small neighbourhood $\tilde{U}_j$ of $\overline{U}_j$. Then the diffeomorphism Φ defined by the flow ϕ^h_1 is a symplectomorphism of $\mathbb{C}^2$ which is the identity map outside $\tilde{U}_j$. By construction, $\Phi \circ \iota$ is a standard open Whitney umbrella near p_j. Repeating this procedure for all umbrella points gives a new Lagrangian inclusion (denoted again by ι) with only standard umbrellas. Thus we obtain the following version of Givental's theorem.

Proposition 2.1. *Let S be a compact real surface without boundary. There exists a Lagrangian inclusion $\iota : S \to \mathbb{C}^2$ such that all its open Whitney umbrella points are standard. Furthermore, if $S \neq S^2$ or $\mathbb{R}P_2$, then S admits a singular Lagrangian embedding with only standard umbrellas and without double points.*

3. Rational Convexity of Lagrangian inclusions

Here we prove the following

Proposition 3.1. *Let S be a compact real surface without boundary and let $\iota : S \mapsto (\mathbb{C}^2, \omega_{\rm st})$ be a Lagrangian inclusion given by Proposition 2.1. Then $\iota(S)$ is rationally convex in $\mathbb{C}^2$.*

Proposition 3.1 was already proved by the authors [17] in the special case of a Lagrangian inclusion with a single umbrella. We include here a detailed presentation for convenience of the reader.

We will identify S and $\iota(S)$ as a slight abuse of notation. The ball of radius ε centred at a point p is denoted by $\mathbb{B}(p, \varepsilon)$, and the standard Euclidean distance between a point $p \in \mathbb{C}^n$ and a set $Y \subset \mathbb{C}^n$ is denoted by ${\rm dist}(p, Y)$. Our approach is a modification of the method of Duval-Sibony and Gayet. The main tool here is the following result.

Lemma 3.2 ([8], [9]). *Let ϕ be a plurisubharmonic C^∞-smooth function on $\mathbb{C}^n$, and let h be a C^∞-smooth function on $\mathbb{C}^n$. Let $X = \{|h| = e^\phi\}$ be compact. Suppose that*

(1) $|h| \le e^\phi$;
(2) $\overline{\partial} h = O({\rm dist}(\cdot, S)^{\frac{3n+5}{2}})$;
(3) $|h| = e^\phi$ *with order at least 1 on S;*
(4) *For any point $p \in X$ at least one of the following conditions holds: (i) h is holomorphic in a neighbourhood of p, or (ii) p is a smooth point of S, and ϕ is strictly plurisubharmonic at p.*

Then X is rationally convex.

We remark that if follows from the proof of the lemma in [9] that in fact, we may assume that ϕ is merely continuous at points where h is holomorphic.

The proof of Proposition 3.1 consists of finding the functions ϕ and h that satisfy Lemma 3.2 with $X = S$. This will be achieved in three steps: we first construct a closed $(1,1)$-form ω that vanishes near singular points of S and such that $\omega|_S = 0$. The form ω is a modification of the standard symplectic form ω_{st} in $\mathbb{C}^2$ near singular points of S. Near self-intersection points this is done in the paper of Gayet [9], and so we will deal with the umbrella points. Secondly, from ω and its potential ϕ we construct the required function h. In the last step we replace ϕ with a function $\phi + \rho$, for a suitable ρ, so that the pair $\{\phi + \rho, h\}$ satisfies all the conditions of Lemma 3.2.

Step 1: the form ω. Our modification of the form ω_{st} and its potential is an inductive procedure on the umbrella points. Let $p_1, \dots, p_m$ be the umbrella points on S, $p_j = (x_j, u_j, y_j, v_j)$. By the assumption in Theorem 3.1, after a translation of p_j to the origin, the surface S is parametrized near p_j by the mapping π given by (1). Let $L_j : (z, w) \to (z, w) - p_j$ be the translation of p_j to the origin, so that $\pi_j = L_j^{-1} \circ \pi$ parametrizes S near p_j.

For a function f we have $d^c f = -f_y dx + f_x dy - f_v du + f_u dv$. Using this we have $\pi^* d^c \phi_{\text{st}} = -2t^2 s dt - \frac{2}{3} t^3 ds$. Consider the pluriharmonic function $\zeta_1 = \frac{v^2}{2} - \frac{u^2}{2}$. Then $\pi^* d^c \zeta_1 = \pi^* d^c \phi_{\text{st}}$. The function $\phi_{\text{st}} - \zeta_1$ is strictly plurisubharmonic and satisfies

$$\pi^* d^c (\phi_{\text{st}} - \zeta_1) = 0. \tag{6}$$

Let $\phi_1 = (\phi_{st} - \zeta_1) \circ L_1$. Since L_j are $\mathbb{C}$-linear, they commute with d^c. Therefore, $d^c \phi_1 | S = 0$ near p_1 and $dd^c \phi_1 = \omega_{st}$. Let $r : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth increasing convex function such that $r(t) = 0$ when $t \le \varepsilon_1$ and $r(t) = t - c$ when $t > \varepsilon_2$, for some suitably chosen $c > 0$ and $0 < \varepsilon_1 < \varepsilon_2$. We choose $\varepsilon_2 > 0$ so small that the set $\{\phi_1 < \varepsilon_2\}$ does not contain any singular points of S except p_1. Let

$$\tilde{\phi}_1 = r \circ \phi_1, \quad \omega_1 = dd^c(\tilde{\phi}_1). \tag{7}$$

Then $\pi^* \omega_1 = 0$ by (6). Therefore, the surface S remains Lagrangian with respect to the form ω_1. This gives us the required modification of ω_{st} near p_1. Note that our construction gives two neighbourhoods $U_1 \Subset U_1'$ of p_1, which can be chosen arbitrarily small, so that $\omega_1|_{U_1} = 0$ and $\omega_1 = \omega_{\text{st}}$ in $\mathbb{C}^2 \setminus U_1'$. On the other hand, the potential $\tilde{\phi}_1$ is a global modification of ϕ_{st} but it remains plurisubharmonic on $\mathbb{C}^2$.

Consider now the modification of $\tilde{\phi}_1$ and ω_1 near p_2. Up to an additive constant the potential $\tilde{\phi}_1$ for ω_1 near p_2 agrees with $(\phi_{st} - \zeta_1) \circ L_1$. We construct ϕ_2 in the form

$$\phi_2 = (\tilde{\phi}_1 - \zeta_2) \circ L' + C,$$

with a suitable choice of a function ζ_2 and a constant C. The condition $\pi_2^* d^c \phi_2 = 0$ is equivalent to

$$\pi^* d^c \left((\phi_s t - \zeta_1) \circ L_1 - \zeta_2 \right) = 0.$$

This can be achieved by choosing

$$\zeta_2 = -2x_1 x - 2y_1 y - v_1 v - 3u_1 u.$$

Then $d^c \phi_2 | S = 0$ near p_2. Further, $\phi_2(p_2) = 0$ by a suitable choice of the constant C, and $dd^c \phi_2 = \omega_1$. Now take $\tilde{\phi}_2 = r \circ \phi_2$, where r is as above, and set $\omega_2 = dd^c \tilde{\phi}_2$. This gives the required modification near p_2.

This procedure can be repeated for all other p_j, $j = 2, \dots, m$. Note that at each step the modification of the function $\tilde{\phi}_{j-1}$ is obtained by adding linear terms in (x, u, y, v) precomposed with a translation. This ensures that the form ω_j remains unchanged in the complement of some small neighbourhood U_j' of the point p_j. For the same reason, the function $\tilde{\phi}_j$ remains globally plurisubharmonic, which is, in fact, strictly plurisubharmonic outside the union of the

neighbourhoods U'_j. We repeat this procedure m times for all umbrella points to obtain the function $\tilde{\phi}$ and the form $\tilde{\omega}$.

Denote by $p_{m+1}, \dots, p_N$ the double points of S. Then [9, Prop. 1] gives further modification of the form $\tilde{\omega}$ and its potential $\tilde{\phi}$ near the double points. Combining everything together yields the following result.

Lemma 3.3. *Given $\varepsilon > 0$ sufficiently small, there exists a $(1,1)$-form $\tilde{\omega}$ and $0 < \varepsilon' < \varepsilon$ such that*

(i) $\tilde{\omega}|_S = 0$;

(ii) $\tilde{\omega} = \omega$ *on* $\mathbb{C}^2 \setminus \left(\cup_{j=1}^N \mathbb{B}(p_j, \varepsilon) \right)$.

(iii) $\tilde{\omega}$ *vanishes on* $\mathbb{B}(p_j, \varepsilon')$, $j = 1, \dots, N$.

Furthermore, there exists a smooth function $\tilde{\phi}$ on $\mathbb{C}^2$ such that $dd^c\tilde{\phi} = \tilde{\omega}$. The function $\tilde{\phi}$ is plurisubharmonic on $\mathbb{C}^2$, and strictly plurisubharmonic on $\mathbb{C}^2 \setminus \left(\cup_{j=1}^N \mathbb{B}(p_j, \varepsilon) \right)$.

Step 2: the function h. Let $\iota : S \to \mathbb{C}^2$ be a Lagrangian inclusion, and $\tilde{\phi}$ be the potential of the form $\tilde{\omega}$ given by Lemma 3.3. For simplicity we drop tilde from the notation. We recall the construction in [8] and [9] of a smooth function h on $\mathbb{C}^2$ such that $|h|\,\big|_S = e^{\phi}$ and $\overline{\partial} h(z) = O(\mathrm{dist}(z, S)^6)$.

Let $\tilde{S}$ be a deformation retract of S. Note that it exists because near an umbrella point the surface S is the graph of a continuous vector-function. Let γ_k, $k = 1, \dots, l$, be the basis in $H_1(\tilde{S}, \mathbb{Z}) \cong H_1(S, \mathbb{Z})$ supported on S. Using de Rham's theorem one can find closed forms β_k on $\tilde{S}$ such that $\int_{\gamma_\nu} \beta_k = \delta_{\nu k}$, and such that β_k vanish in the balls $\mathbb{B}(p_j, \varepsilon)$ as in Lemma 3.3 around the singularities of S. Further, there exist smooth functions ψ_k with compact support in $\tilde{S}$ such that ψ_k vanish on $S \cup (\cup_{j=1}^N \mathbb{B}(p_j, \varepsilon))$, and for $k = 1, \dots, l$,

$$\iota^* d^c \phi_k = \iota^* \beta_k. \tag{8}$$

Indeed, for each k, we set $\phi_k = A(z,w) r_1 + B(z,w) r_2$, where $r_1(z,w)$ and $r_1(z,w)$ are local defining functions of S and A, B are some unknown functions. Plugging this expression into (8) gives a linear system for the restrictions of A and B to S that can be solved. A suitable extension of this solution with support in $\tilde{S}$ gives the result. Note that near singular points the extension is identically zero.

For $\lambda_k > 0$ the function $\phi + \sum_{j=1}^l \lambda_k \psi_k$ agrees with ϕ on S. For sufficiently small λ_k it is strictly plurisubharmonic outside the balls $\mathbb{B}(p_j, \varepsilon)$ and globally plurisubharmonic since the functions ψ_k vanish in $\mathbb{B}(p_j, \varepsilon)$. Further, there exists a choice of λ_k and $M > 0$ such that for the function

$$\tilde{\phi} = M \left(\phi + \sum_{j=1}^{l} \lambda_j \psi_j \right) \tag{9}$$

the form $\iota^* d^c \tilde{\phi}$ is closed on S and has periods which are multiples of 2π. Then there exists a C^∞-smooth function $\mu : S \to \mathbb{R}/2\pi\mathbb{Z}$ that vanishes on the intersection of S with $\mathbb{B}(p_j, \varepsilon)$, $j = 1, \dots, N$, and such that $\iota^* d^c \tilde{\phi} = d\mu$. By [12], there exists a function h defined on $\mathbb{C}^2$ such that

$$h|_S = e^{\tilde{\phi} + i\mu}|_S$$

and $\overline{\partial} h(z) = O(\mathrm{dist}(z, S)^6)$. It follows that $\tilde{\phi} - \log|h|$ vanishes to order 1 on S. Note that h is constant near singular points of S. Finally, the function h can be suitably extended to $\mathbb{C}^2$ preserving the inequality given by (1) in Lemma 3.2.

Step 3: the function ϕ. A closed subset K in $\mathbb{C}^n$ is called *locally polynomially convex* near a point $p \in K$ if for every sufficiently small $\varepsilon > 0$ the intersection $K \cap \overline{\mathbb{B}(p, \varepsilon)}$ is polynomially convex in $\mathbb{C}^n$. Again, for simplicity of notation we denote by ϕ the function (9) constructed in Step 2. It does not yet satisfy the conditions of Lemma 3.2 because there are still some smooth points on S where the function h is not holomorphic and ϕ is not strictly plurisubharmonic. For this we will replace ϕ by a function $\tilde{\phi} = \phi + c \cdot \rho$, where the function ρ will be constructed using local polynomial convexity of S, and $c > 0$ will be a suitable constant.

We recall our result from [15, 16].

Lemma 3.4. *Let S be a Lagrangian inclusion in $\mathbb{C}^2$, and let $p_0, \dots, p_N$ be its singular points. Suppose that S is locally polynomially convex near every singular point. Then there exists a neighbourhood Ω of S in $\mathbb{C}^2$ and a continuous non-negative plurisubharmonic function ρ on Ω such that $S \cap \Omega = \{p \in \Omega : \rho(p) = 0\}$. Furthermore, for every $\delta > 0$ one can choose $\rho = (\mathrm{dist}(z, S))^2$ on $\Omega \setminus \cup_{j=1}^N \mathbb{B}(p_j, \delta)$; in particular, it is smooth and strictly plurisubharmonic there.*

The standard open Whitney umbrella is locally polynomially convex by [15], and S is locally polynomially convex near transverse double self-intersection points by [16]. For the proof of the lemma we refer the reader to [16].

To complete the construction of the function ϕ, we choose the function ρ in Lemma 3.4 with $\delta > 0$ so small that the balls $\mathbb{B}(p_j, \delta)$ are contained in balls $\mathbb{B}(p_j, \varepsilon'/2)$ given by Lemma 3.3. Note that ρ is defined only in a neighbourhood Ω of S, but we can extend it as a smooth function with compact support in $\mathbb{C}^2$. Consider now the function

$$\tilde{\phi} = \phi + c \cdot \rho.$$

We choose the constant $c > 0$ so small that the function $\tilde{\phi}$ remains to be plurisubharmonic on $\mathbb{C}^2$. At the same time, since $c > 0$ and ρ is strictly plurisubharmonic on S outside small neighbourhoods of singular points, we conclude that the function $\tilde{\phi}$ is strictly plurisubharmonic outside the balls $\mathbb{B}(p_j, \delta)$. It also follows that $X = \{|h| = e^{\tilde{\phi}}\} = S$. The pair $\tilde{\phi}$ and h now satisfies all the conditions of Lemma 3.2. This completes the proof of Proposition 3.1.

For the proof of Theorem 1.1 we will also need the following result.

Corollary 3.5. *Suppose that $\iota : S \to \mathbb{C}^2$ is a Lagrangian inclusion of a compact surface. Then $\iota(S)$ admits a Stein neighbourhood basis.*

Indeed, one can take neighbourhoods of $\iota(S)$ of the form $\{\rho < \varepsilon\}$ where ρ is a function given by Lemma 3.4 and $\varepsilon > 0$ is small enough.

4. Rational approximation on surfaces

The classical Oka-Weil theorem (see, e.g., [18]) states that any holomorphic function in a neighbourhood of a rationally convex compact set $X \subset \mathbb{C}^n$ can be approximated uniformly on X by rational functions with poles off X. Rational functions can be replaced by holomorphic polynomials if X is polynomially convex. We will need the following approximation result, which is due to O'Farrel-Preskenis-Walsch [14] (see also Stout [18]):

Let X be a compact holomorphically convex set in $\mathbb{C}^n$, and let X_0 be a closed subset of X for which $X \setminus X_0$ is a totally real subset of the manifold $\mathbb{C}^n \setminus X_0$. A function $f \in C(X)$ can be approximated uniformly on X by functions holomorphic on an neighbourhood of X if and only if $f|_{X_0}$ can be approximated uniformly on X_0 by functions holomorphic on an neighbourhood of X.

Recall that a set X is called a *totally real set* of a manifold $\mathcal{M}$ if there is a neighbourhood U of X in $\mathcal{M}$ on which is defined a nonnegative strictly plurisubharmonic function ϕ of class C^2 such that $X = \{p \in U : \phi(p) = 0\}$. The following result can be found in Stout [18, Thm 6.2.9]:

A compact connected subset X of a Stein manifold $\mathcal{M}$ is holomorphically convex if and only if there is a sequence Ω_j of domains in $\mathcal{M}$ with $\Omega_j \supset \Omega_k$, when $j \le k$, and with $\bigcap_j \Omega_j = X$ such that if for each j, $(\tilde{\Omega}_j, \mathrm{proj}_j)$ is the envelope of holomorphy of Ω_j, then $\bigcap_j \mathrm{proj}_j(\tilde{\Omega}_j) = X$.

Suppose now that $X = \iota(S)$ is a Lagrangian inclusion given by Proposition 2.1; it is rationally convex by Proposition 3.1. Let X_0 be the set of singular points of X, i.e., the set of double points and Whitney umbrellas. Then $X \setminus X_0$ is a smooth totally real submanifold, and so for each point $p \in X \setminus X_0$ there exists a neighbourhood in which the square of the distance to X is a strictly plurisubharmonic function. From these neighbourhoods we can construct a neighbourhood $U \supset X \setminus X_0$ with a nonnegative strictly plurisubharmonic function on it that vanishes on $X \setminus X_0$. This shows that $X \setminus X_0$ is a totally real set in $\mathbb{C}^2 \setminus X_0$.

The set X_0 is finite, hence it satisfies the assumption of of O'Farrel-Preskenis-Walsch theorem. By Lemma 3.5, $X \subset \mathbb{C}^2$ admits a Stein neighbourhood basis $\{\Omega_j\}_j$. Each Ω_j is Stein, therefore, $\tilde{\Omega}_j = \Omega_j$, and it follows from above that X is holomorphically convex. Thus, all conditions in the result of O'Farrel-Preskenis-Walsch, stated above, are satisfied, and we conclude that any continuous function on X can be approximated by holomorphic functions in a neighbourhood of X, hence by rational functions as seen by the Oka-Weil theorem. Combining everything together gives the following.

Proposition 4.1. *If $\iota : S \to \mathbb{C}^2$ is a Lagrangian inclusion with standard umbrellas, then any continuous function on $\iota(S)$ can be approximated uniformly on $\iota(S)$ by rational functions with poles off $\iota(S)$.*

With this the main result is easily verified.

Proof of Theorem 1.1. (i) and (ii). By Proposition 2.1, there exists a singular Lagrangian embedding $f = (f_1, f_2) : S \to \mathbb{C}^2$ with standard umbrellas as the only singularities. The required statements now follow from Proposition 4.1.

(iii) Formula (4) gives an immersion of the sphere S^2 into $\mathbb{C}^2$ with one double point, but this does not give the approximation result because the coordinate functions attain the same value at the double point. However, by the Borsuk-Ulam theorem (see, e.g., [11]), any continuous function $F : S^2 \to \mathbb{R}^2$ has at least two antipodal points p and q on S^2 where it attains the same value. Hence, it can be approximated by rational functions but only after we apply a rotation of S^2 that sends p and q to the north and south poles of S^2, which are the preimages of the double point.

(iv) A similar story holds for $\mathbb{R}P_2$, for which one needs two double points. Let $f = (f_1, f_2) : \mathbb{R}P_2 \to \mathbb{C}^2$ be the Lagrangian inclusion with two double points and one standard umbrella. By the Whitney approximation theorem it suffices to approximate any smooth function $F : \mathbb{R}P_2 \to \mathbb{C}$. Since $\mathbb{R}P_2$ cannot be diffeomorphic to any subset of $\mathbb{C}$, a generic point in the image of F will have at least two pre-images. Applying a diffeomorphism τ of $\mathbb{R}P_2$ we may assume that there exist points $p_j, q_j \in \mathbb{R}P_2$ such that $(F \circ \tau)(p_j) = (F \circ \tau)(q_j)$, $j = 1, 2$, and $f_j(p_k) = f_j(q_k)$, $j, k = 1, 2$. Then by Proposition 4.1, $F \circ \tau$ can be approximated by rational combinations of f_1 and f_2. $\square$

References

[1] H. Alexander and J. Wermer. Several complex variables and Banach algebras. Third edition. Graduate Texts in Mathematics, 35. Springer-Verlag, New York, 1998. xii+253 pp. ISBN: 0-387-98253-1.

[2] V. Arnold. *Singularities of ray systems.* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 27-49, PWN, Warsaw, 1984.

[3] V. Arnold. *Symplectic geometry and topology.* J. Math. Phys. **41** (2000), no. 6, 3307-3343.

[4] M. Audin. *Quelques remarques sur les surfaces lagrangiennes de Givental.* JGP. Vol 7, n. 4 (1990) p. 583–598.

[5] K. Cieliebak and Y. Eliashberg. *The topology of rationally and polynomially convex domains.* Invent. Math. **199** (2015), no. 1, 215–238.

[6] J. Duval, *Convexité rationelle des surfaces lagrangiennes*, Invent. math. **104** (1991), 581-599.

[7] J. Duval and D. Gayet. *Rational convexity of non-generic immersed Lagrangian submanifolds.* Math. Ann. **345** (2009), no. 1, 25–29.

[8] J. Duval and N. Sibony. *Polynomial convexity, rational convexity, and currents.* Duke Math. J., Vol 79, No. 2 (1995), 487–513.

[9] D. Gayet. *Convexité rationnelle des sous-variétés immergées Lagrangiennes.* Ann. Sci. Ecole Norm. Sup. (4) **33** (2000), no. 2, 291–300.

[10] A. B. Givental, *Lagrangian imbeddings of surfaces and the open Whitney umbrella.* Funktsional. Anal. i Prilozhen. **20** (1986), no. 3, 35–41, 96.

[11] A. Hatcher, Algebraic topology. Cambridge University Press, Cambridge, 2002. xii+544 pp. ISBN: 0-521-79160-X; 0-521-79540-0.

[12] L. Hörmander and J. Wermer. *Uniform approximation on compact sets in* $\mathbb{C}^n$. Math. Scand. **23** (1968), 5–21.

[13] S. Nemirovski and K. Siegel. *Rationally Convex Domains and Singular Lagrangian Surfaces in* $\mathbb{C}^2$. To appear in Invent. Math., preprint available at http://arxiv.org/abs/1410.4652 .

[14] A.G. O'Farrell, K.J. Preskenis, and D. Walsh. *Holomorphic approximation in Lipschitz norms.* Proceedings of the conference on Banach algebras and several complex variables (New Haven, Conn., 1983), p. 187–194, Contemp. Math., **32**, Amer. Math. Soc., Providence, RI, 1984.

[15] R. Shafikov and A. Sukhov. *Local polynomial convexity of the unfolded Whitney umbrella in* $\mathbb{C}^2$. IMRN, **22**, 5148–5195.

[16] R. Shafikov and A. Sukhov. *Polynomially convex hulls of singular real manifolds.* Trans. Amer. Math. Soc., **368** (2016), 2469-2496.

[17] R. Shafikov and A. Sukhov. *Lagrangian inclusion with an open Whitney umbrella is rationally convex.* arXiv: 1502.05586v1, to appear in Contemp. Math.

[18] E. L. Stout, Polynomial convexity. Progress in Mathematics, 261. Birkhäuser Boston, Inc., Boston, MA, 2007. xii+439 pp.